problem. Hence, solution of (3.25) can also be regarded as computation of the optimal correction in the case when only one measurement is possible during motion. It is expedient to make this measurement immediately before correction, i.e. at the instant τ given by Eqs. (3.25).

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ON THE SOLUTION OF VARIATIONAL PROBLEMS OF SUPERSONIC FLOWS OF GAS WITH FOREIGN PARTICLES

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Questions arising in solving the problem of design of the optimum contour of the supersonic portion of plane and axisymmetric nozzles for flows involving any nonequilibrium processes are considered. An investigation is carried out on the example of the flow of gas containing foreign particles (solid, or liquid) by using Lagrange multipliers in the form first applied to problems of supersonic gas dynamics by Guderley and Armitage [1].

The exactly formulated problem of design of the supersonic portion of plane and axisymmetric nozzles for nonequilibrium flows were considered in papers [2 and 3], while papers [4 and 5] dealt with the problem of flow of gas with foreign particles. In deriving the conditions necessary for the determination of the optimum authors of these papers had considered that case only in which the first set characteristic bounding on the right the region of influence of the sought contour intersects the rarefaction wave beam closing characteristic originating in the flow past the starting point of a (contour) kink, or in the case of a curvature constraint in the flow past the initial section of the maximum permissible curvature (*). The consideration of that case only appeared natural, as for

*) As will be clear from the following, in this case in the system of conditions derived in [4 and 5] the conditions along the particle streamline separating the region containing particles from the particle-free gas have been lost.

stabilized flows the given configuration was the only one possible [6]. Solution of variational problems in univariate approximation [7–10] had at the same time shown that with nonequilibrium flows an increase of the nozzle optimum length for a given back pressure always results in an increase of its thrust, contrary to the equilibrium case in which there exists a certain nozzle length beyond which the thrust ceases to increase. From the solution of the variational problem of the optimum slender profile for nonequilibrium flows [11] it follows moreover that nonequilibrium leads to the decrease of the kink magnitude at its starting point. These conclusions are in agreement with physical concepts of the relation between thrust losses and deviation from equilibrium. It appears therefore expedient to consider as a possible optimal configuration, besides the scheme investigated in [2–5], the contour the closing characteristic of which begins at the axis of symmetry outside of the rarefaction wave beam. It appears that if such a contour is optimal, it must have inner kink points.

1. Using the model of a two-velocity two-temperature continuous medium as an approximation, we shall consider a stationary or axisymmetric flow of a mixture of gas and foreign particles (solid or liquid). Let x, y be orthogonal coordinates with the x-axis directed from left to right along the axis of symmetry, ρ the density, p the pressure, h the specific enthalpy, $\mathbf{V} = (u, v)$ the gas velocity vector, ρ_s the mean density, T_s the temperature, $e_s = e_s(T_s)$ the specific inner energy, $\mathbf{V}_s = (u_s, v_s)$ the "gas-particle" velocity vector, $\mathbf{f} = (f_x, f_y)$ the gas and particles interaction force, and q the heat flux between these both related to the particle unit mass, with $\nu = 0$, or 1 for the plane and axisymmetric cases respectively. Then, in the absence of phase transfers, external forces and heat sources, and neglecting the volume of particles, the flow here considered will be defined by Eqs. (see, e.g., [12])

$$L_1 \equiv u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\rho_s}{\rho} f_x = 0$$

$$L_2 \equiv u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\rho_s}{\rho} f_y = 0$$

$$L_3 \equiv \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial x} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \frac{\nu\rho v}{y} = 0$$

$$L_4 \equiv u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} - \frac{u}{\rho}\frac{\partial p}{\partial x} - \frac{v}{\rho}\frac{\partial p}{\partial y} + N = 0$$

$$L_5 \equiv u_s\frac{\partial u_s}{\partial x} + v_s\frac{\partial u_s}{\partial y} - f_x = 0, \quad L_6 \equiv u_s\frac{\partial v_s}{\partial x} + v_s\frac{\partial v_s}{\partial y} - f_y = 0 \qquad (1.1)$$

$$L_7 \equiv \frac{\partial u_s}{\partial x} + \frac{\partial v_s}{\partial y} + \frac{u_s}{\rho_s}\frac{\partial \rho_s}{\partial x} + \frac{v_s}{\rho_s}\frac{\partial \rho_s}{\partial y} + \frac{\nu v_s}{y} = 0$$

$$L_8 \equiv u_s\frac{\partial e_s}{\partial x} + v_s\frac{\partial e_s}{\partial y} - q = 0 \quad \left(N = \frac{\rho_s}{\rho}[(\mathbf{V}_s - \mathbf{V})\mathbf{f} + q]\right)$$

System (1.1) is completed by the expressions of h, $\mathbf{f}$ and q which we shall write in the form

$$h = h(p, \rho), \quad \mathbf{f} = \mathbf{f}(p, \rho, u, v, u_s, v_s, T_s), \quad q = q(p, \rho, u, v, u_s, v_s, T_s)$$

This assumes in particular a thermodynamic equilibrium of the gas (but not between the gas and particles).

For $w \equiv |\mathbf{V}| > a$, where a is the velocity of sound in the gas, and is expressed by:

$$a^{-2} = (1 - \rho h_p) / \rho h_\rho \qquad (\omega_p = (\partial\omega / \partial p)_\rho, \ \omega_\rho = (\partial\omega / \partial\rho)_p)$$

system (1.1) has four sets of real characteristics. These are the gas streamlines along which

$$vx' - u = 0, \quad \rho vh' - vp' + \rho N = 0, \quad v\,[p' + \rho(w^2/2)'] + \rho_s \mathbf{f}\,\mathbf{V} = 0 \qquad (1.2)$$

the particle streamlines where

$$v_s x' - u_s = 0, \quad v_s (w_s^2)' - 2\mathbf{f}\mathbf{V}_s = 0, \quad v_s e_s' - q = 0 \quad (w_s = |\mathbf{V}_s|) \qquad (1.3)$$

and two sets of Mach lines satisfying Eqs.

$$x' = \frac{-uv \pm a\sqrt{w^2 - a^2}}{a^2 - v^2}$$

$$\frac{a^2 + u(vx' - u)}{\rho x'} p' + a^2 vu' + \frac{uv'}{x'^2}[a^2 - (vx' - u)^2] +$$

$$+ a^2(vx' - u)\left(\frac{\nu v}{y} - \frac{\rho_s}{\rho a^2}\mathbf{f}\mathbf{V} - \frac{N}{\rho h_p}\right) + \frac{\rho_s}{\rho}[a^2 + u(vx' - u)]\left(f_x + \frac{1}{x'} f_y\right) = 0 \qquad (1.4)$$

Primes in (1.2) — (1.4) denote derivatives with respect to y taken in the appropriate direction, and the upper (lower) sign in (1.4) defines the direction of characteristics of the first (second) set.

2. Let it be required to define the nozzle contour ag which would ensure the maximum of thrust χ for a known supersonic flow at entry, i.e. to the left of a certain characteristic ac of the second set, as shown on Fig. 1, where characteristics of the first and second sets are drawn in thin lines. The nozzle maximum length X, its surface area, etc. may be prescribed in addition.

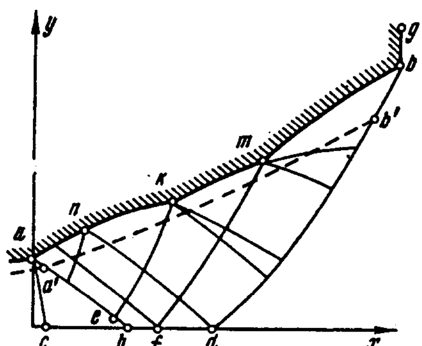

Fig. 1

Generally, for reasons stated in the previously considerd problems, there exists at point a a kink (absence of curvature constraints is assumed), and if the (nozzle) length is limited, the optimal contour may be provided with an end face bg where $x \equiv X$.

We shall confine ourselves to the case in which at point a a flow past a convex angle is realized, with the pressure p^+ on the presumed end face constant and independent of the unknown contour form (absence of flow of gas past the end face, or its part during the variation of these is assumed). Then, with an approximation to the negligibly small terms and factors

$$\chi = \int_a^b y^\nu p\,dy + \int_b^g y^\nu p^+ dy$$

and the possible m isoperimetric conditions will be presented in the form

$$K^j = \int_a^b \varphi^j(y, x, x', v, p, \rho)\,dy + \int_b^g \varphi^{oj}(y, x, x')\,dy \quad (j = 1, \ldots, m) \qquad (2.1)$$

where K^j are given constant, and φ^j and φ^{oj} known functions; subscripts a, b... (except subscript s) denote parameters at corresponding points, and integration is carried out along the nozzle contour.

Due to particle lag a layer of particle-free gas is usually formed along the wall [13 and 14], and this has been taken into account in (2.1). In principle particles may still reach the wall, which generally is undesirable. Taking this into account we shall consider out of all possible contours defined by Eq. $x = \xi(y)$ only those for which

$$\zeta(y) - \xi(y) \geqslant H > 0 \quad \text{for} \quad y_a \leqslant y \leqslant y_b \tag{2.2}$$

where $x = \zeta(y)$ is the equation of the particle streamline $a'b'$ bounding from below the particle-free gas, as shown on Fig. 1 by the dotted line, and H is either given constant, or a known function of y.

3. In order to solve the problem we compile the functional

$$I = \int_a^b \left[\Phi + \alpha\left(\xi' - \frac{u}{v}\right)\right] dy + \int_b^g F dy + \int_{a'}^{b'} \rho_s [\beta_1 (v_s\zeta' - u_s) + \beta_2 L_5 +$$

$$+ \beta_3 L_6 + \beta_4 L_8]\, dy + \iint_G \left(\sum_{k=1}^{4} \mu_k L_k + \rho_s \sum_{k=5}^{8} \mu_k L_k\right) dx\, dy$$

$$\Phi(y, x, x', v, p, \rho, \lambda) = y^\nu p + \sum_{j=1}^{m} \lambda_j \varphi^j (y, x, x', v, p, \rho)$$

$$F(y, x, x', \lambda) = y^\nu p^+ + \sum_{j=1}^{m} \lambda_j \varphi^{oj} (y, x, x')$$

Here λ_j are constants $\alpha = \alpha(y)$, $\beta_i = \beta_i(y)$ and $\mu_i = \mu_i(x, y)$ are variable Lagrange muliplіers, and G the sought contour influence region generally bounded by the last characteristic of beam ah, the axis of symmetry hd, the first set characteristic db and the contour ab itself.

For permissible variations the variations in I and χ coincide by virtue of (1.1) – (1.3) and (2.1).

The conditions necessary for maximum χ defining the optimal contour form are obtained from the analysis of the first variation $\delta\chi = \delta I$. Such countours ab may in accordance with (2.2) consist of sections of two kinds (Fig. 1), viz., sections ak and mb where

$$z \equiv \zeta(y) - \xi(y) - H > 0 \tag{3.1}$$

and the nozzle contour may be arbitrarily varied, and also of sections (one of such sections km, is shown on Fig. 1) where

$$z \equiv \zeta(y) - \xi(y) - H = 0 \tag{3.2}$$

and the only permissible variations are those which increase the distance between the wall and the boundary streamline $a'b'$, i. e. such that $\delta z \geqslant 0$. Junction points of the various contour sections will generally be kink points. Segments satisfying (3.1) may also connect at the kink points. Boundary line $a'b'$, as well as other particle streamlines along which $x' = u_s/v_s$ have by virtue of the particle equations of motion less than two continuous derivatives (ζ' and ζ'') when $H > 0$.

The variation of I is carried out in conformity with [1, 2 and 15]. Special consideration must be given to variation in kink point positions [15], and lines of discontinuity of the Lagrange multipliers are permitted [2. and 15]. When varying the ends of sections (3.2) translations in the direction of $x' = \zeta'(y)$ only are arbitrary, while translations in other directions are bounded by condition $\delta z \geqslant 0$. Because of the absence of particles along ab it is furthermore considered that $\delta p = -\rho u \delta u - \rho v \delta v$, $\delta\rho = a^{-2}\, \delta p$.

4. The selection of Lagrange multipliers is made in such a way as to obtain the disappearance of all variations in $\delta\chi$, except $\delta\xi$ along sections (3.1) and δz along sections (3.2), and of the kink point coordinate increments. Such a selection is possible for any arbitrary contour (not necessarily the optimum one), and leads to equations with boundary conditions which define the Lagrange multipliers.

The equations of $\mu_1, \ldots, \mu_4$ in their continuity subregions of the influence area $abdha$ are equivalent to the following differential relationships:

$$u\mu_1' + v\mu_2' + \rho\mu_3' - \mu_1\left(u' + \frac{\nu u}{y}\right) - \mu_2\left(v' + \frac{\nu v}{y}\right) - \left(\mu_1\frac{u}{\rho} + \mu_2\frac{v}{\rho} + \right.$$
$$\left. + \mu_4 h_\rho\right)\rho' + \mu_4\frac{h_\rho}{a^2}p' - \mu_3\frac{\nu\rho}{y} + \mu_4\frac{\rho_s}{\rho}(x'f_x + f_y) - \rho_s[x'P(u) + P(v)] = 0$$
$$v\mu_3' + vh_\rho\mu_4' - \mu_1\frac{v}{\rho}u' - \mu_2\frac{v}{\rho}v' - \mu_4\frac{v}{\rho^2}p' -$$
$$- \mu_4\frac{vh_\rho}{\rho}\rho' - \mu_3\frac{\nu v}{y} - \mu_4 h_\rho\frac{\nu v}{y} + \mu_4\frac{N}{\rho} - \rho_s P(\rho) = 0 \qquad (4.1)$$

fulfilled along the gas streamlines (1.2), and to two equations corresponding to the two values of x' in (1.4)

$$\frac{vx' - u}{\rho}v\mu_1' + \frac{u - vx'}{\rho}u\mu_2' + (v + ux')\mu_3' - \mu_1\left(\frac{ux' + v}{\rho}u' + \right. \qquad (4.2)$$
$$\left. + \frac{vx' - u}{\rho^2}v\rho' + \frac{vx' - u}{y\rho}\nu v\right) - \mu_2\left(\frac{ux' + v}{\rho}v' - \frac{vx' - u}{\rho^2}u\rho' - \right.$$
$$\left. - \frac{vx' - u}{y\rho}\nu v x'\right) - \mu_3\frac{\nu}{y}(ux' + v) - \mu_4\left\{\frac{ux' + v}{\rho}h_\rho\rho' + \frac{vx' - u}{a^2}v^2 h_\rho\left(\frac{u}{v}\right) + \right.$$
$$\left. + \frac{1 + x'^2}{y}\nu v h_\rho - \frac{\rho_s}{\rho}h_p(vx' - u)(f_x - x'f_y) - \left[1 + x'^2 - \frac{(vx' - u)^2}{\rho h_\rho}\right]\frac{N}{\rho}\right\} -$$
$$- \frac{\rho_s}{\rho}(vx' - u)[P(u) - x'P(v) + \rho(vx' - u)P(p)] - \rho_s(1 + x'^2)P(\rho) = 0$$

which hold along the characteristics of the first and second sets of (1.4). Primes in (4.1) and (4.2) denote as before the total derivatives with respect to y taken along corresponding directions, and

$$P(r) = \frac{\mu_1}{\rho}\frac{\partial f_x}{\partial r} + \frac{\mu_2}{\rho}\frac{\partial f_y}{\partial r} + \frac{\mu_4}{\rho}\left[(u_s - u)\frac{\partial f_x}{\partial r} + \right.$$
$$\left. + (v_s - v)\frac{\partial f_y}{\partial r} + \frac{\partial q}{\partial r}\right] - \mu_5\frac{\partial f_x}{\partial r} - \mu_6\frac{\partial f_y}{\partial r} - \mu_8\frac{\partial q}{\partial r}$$

where r denotes any of the parameters on which $\mathbf{f}$ and q depend.

Multipliers $\mu_5, \ldots, \mu_8$ which introduce into I the equations of particle motion are required in area $a'b'dha'$ only, and are defined here by the system

$$u_s\mu_5' + \mu_7' - \frac{\mu_4}{\rho v_s}\mathbf{V}_s\mathbf{f} - \mu_5\left(u_s' + \frac{u_s}{\rho_s}\rho_s' + \frac{\nu u_s}{y}\right) - \qquad (4.3)$$
$$- \mu_6\left(v_s' + \frac{v_s}{\rho_s}\rho_s' + \frac{\nu v_s}{y}\right) - \mu_7\frac{\nu\rho_s}{y} - \mu_8 e_s' - P(v_s) - x'P(u_s) = 0$$
$$v_s\mu_7' - \frac{1}{\rho}(\mu_1 f_x + \mu_2 f_y) - \mu_4\frac{N}{\rho_s} - \mu_7\frac{\nu v_s}{y} = 0$$
$$v_s C_s\mu_8' - \mu_8 C_s v_s\left(\frac{1}{\rho_s}\rho_s' + \frac{\nu}{y}\right) - P(T_s) = 0$$

$$[\mu_1]_{b'} = [\mu_2]_{b'} = [\mu_3]_{b'} = [\mu_4]_{b'} = 0$$

i.e. multipliers $\mu_1, \ldots, \mu_4$ are continuous when crossing $a'b'$.

Multipliers $\mu_1, \ldots, \mu_4$ may be discontinuous not only along line $a'b'$, the boundary of the two different flow regions in crossing which all of the normal derivatives of gas parameters become discontinuous, but also along certain of the characteristics of the first and second set. If $[\mu_i]$ is the difference of μ_i on the two sides of discontinuity, then along such characteristics

$$x'[\mu_1] + [\mu_2] = 0, \qquad (u - vx')[\mu_1] + \rho[\mu_3] = 0$$
$$[\mu_3] + h_\rho[\mu_4] = 0, \quad [\mu_5] = [\mu_6] = [\mu_7] = [\mu_8] = 0 \tag{4.9}$$

These equalities together with the differential equation which is obtained by substituting in (4 2) for $\mu_1, \ldots, \mu_4$ their jumps, and omitting terms proportional to $\mu_5, \ldots, \mu_8$, completely defines the changes of Lagrange multiplier jumps along the discontinuity lines.

After the necessary integration we obtain

$$[\mu_2]_i = k_i y^{\nu/2} x' \left(\frac{\rho}{(ux'+v)(u-vx')}\right)^{1/2} \exp\left(\int_0^y U\,dy\right) \qquad (i = 1, 2) \tag{4.10}$$

where the constant of integration k_1 (k_2) corresponds to the case in which the discontinuity is a characteristic of the first (second) set, and U is a function of the stream parameters equal to

$$\frac{2\rho(ux'+v)}{\rho_s} U = \frac{\partial f_x}{\partial u} - x'\left(\frac{\partial f_y}{\partial u} + \frac{\partial f_x}{\partial v} - x'\frac{\partial f_y}{\partial v}\right) +$$
$$+ \frac{V_s - V}{h_\rho}\left[\frac{vx'-u}{\rho}\left(x'\frac{\partial f}{\partial v} - \frac{\partial f}{\partial u}\right) - (vx'-u)^2\frac{\partial f}{\partial p} - (1+x'^2)\frac{\partial f}{\partial \rho}\right] +$$
$$+ \rho(vx'-u)\left(\frac{\partial f_x}{\partial p} - x'\frac{\partial f_y}{\partial p}\right) + \frac{1+x'^2}{vx'-u}\left(\rho\frac{\partial f_x}{\partial \rho} - f_x - \rho x'\frac{\partial f_y}{\partial \rho} + x' f_y\right) +$$
$$+ \frac{vx'-u}{\rho h_\rho}\left[f_x - \frac{\partial q}{\partial u} - x' f_y + x'\frac{\partial q}{\partial v} - \rho(vx'-u)\frac{\partial q}{\partial p}\right] -$$
$$- \frac{1+x'^2}{h_\rho}\frac{\partial q}{\partial \rho} + \left[1 + x'^2 - \frac{(vx'-u)^2}{\rho h_\rho}\right]\frac{(V_s - V)f + q}{\rho h_\rho}$$

5. Besides the contour kink points which in accordance with the last condition of (4.4) result in the discontinuity of multipliers $\mu_1, \ldots, \mu_4$ along the characteristics reaching these points (ek and fm in the case shown on Fig. 1) there is yet another cause of discontinuity formation defined by relations (4.9) and (4.10).

We shall prove that, as in the derivation of (4.10), we have along db

$$\mu_2 = k_1 y^{1/2\nu} x' \left(\frac{\rho}{(ux'+v)(u-vx')}\right)^{1/2} \exp\left(\int_0^y U\,dy\right) \tag{5.1}$$

and that at point b in accordance with the first and last of conditions (4.4)

$$k_1 = v_b y_b^{-1/2\nu} \left[\frac{u - vx'}{\rho(ux'+v)}\right]_b^{1/2} \exp\left(-\int_0^{y_b} U\,dy\right)$$

Hence, for example, in the plane case ($\nu = 0$) all along db including point d multiplier μ_2 is also different from zero when $v_b \neq 0$. However, along segment hd of the axis of symmetry we have by virtue of the last but one of conditions (4.4) $\mu_2 = 0$, consequently the characteristic of the second set reaching d is a line of discontinuity of multipliers $\mu_1, \ldots, \mu_4$. Taking into account that along the axis $v = 0$, and that x' for the first and second set characteristics differ as to their signs only, we find from (5.1) and (4.10) that for $\nu = 0$ the discontinuity along "reflected" characteristic of the second set is defined by Formula (4.10)

$$k_2 = k_1 \sqrt{-1} \tag{5.2}$$

if $[\mu_j]$ is taken as the difference of values of μ_j to the left and right of the characteristic.

It is evident from (5.1) that in the axisymmetric case $\mu_{2d} = 0$ when $k_1 \neq 0$, hence, the condition $\mu_2 = 0$ is not violated at point d. It can be shown, however, that while to the left of d the magnitude $\mu_{2x} \equiv \partial\mu_2/\partial x$ is identically zero, in the case of $\nu = 1$ the magnitude μ_{2x} tends to infinity as $y^{-1/2}$ when approaching the axis of symmetry along db. This leads to the conclusion that the reflected characteristic dn is a line of discontinuity of not only of μ_{2x}, but also of the multiplier μ_2 itself. Here, as in the plane case, k_2 in (4.10) is determined from (5.2).

Without going into the details of a rather cumbersome proof of the adduced statements, we shall note that it is based on the use of semicharacteristic variables $y\xi^i$ where ξ^i is a constant along every characteristic of the ith set, with variables $y\xi^1$ used for analyzing the behavior of μ_{2x} along db, and variables $y\xi^2$ along the reflected characteristic.

Condition (5.2) also holds for a discontinuity reflected from the axis of symmetry and arriving, for example, along a characteristic of the first set (as at point f on Fig. 1). In this case it is immaterial whether $[\mu_i]$ is taken as the difference of the left-, and right-hand values of μ_i, or vice versa. It is only important to use the same definition of $[\mu_i]$ for the incoming and the reflected characteristics.

A discontinuity of multipliers reaching the wall along a characteristic of the second set is reflected along the characteristic of the first set. We denote by subscript minus (plus) the flow parameters and the Lagrange multipliers to the left (right) of the reflection point of the wall and, as in (4.10), subscripts 1 and 2 to the magnitudes along the characteristic of the first and second set respectively. If the reflection point n is a point at which ξ' is continuous, then at that point by virtue of (4.8) and (4.9), and of the last of Eqs. (4.4)

$$[\mu_2]_1 = -\frac{x_1'(ux_2'+v)}{x_2'(ux_1'+v)}[\mu_2]_2, \quad \alpha_- - \alpha_+ = \frac{\rho v w^2 (x_2' - x_1')}{x_2'(ux_1'+v)}[\mu_2]_2 \tag{5.3}$$

where all parameters relate to the reflection point. The term $(\alpha_- - \alpha_+)_n \delta x_n$ appears in this case outside of the integral of $\delta\chi$.

The intensity of discontinuity along the characteristic of the first set reaching a kink point is determined from the last equality of (4.4) written for a point of the wall to the left of the angle point for values of μ_i at the upper point of the second set characteristic bounding the rarefaction wave beam on the left, independently of whether it is one of the characteristics of a beam of lines of discontinuity of $\mu_1, \ldots, \mu_4$, or not (boundary characteristics of beams emanating from k and m, as well as other characteristics are shown on Fig. 1 by thin lines).

6. Having selected the Lagrange multipliers in accordance with the equations and conditions derived in the two preceding sections, we obtain for $\delta\chi$ the expression of the form (variations of point g are not considered here)

$$\delta\chi = (U^{(1)}\Delta x + U^{(2)}\Delta y)_b + \sum_j (V^{(1)}\Delta x + V^{(2)}\Delta y)_j +$$

$$+ (\alpha_- - \alpha_+)_n \delta x_n + \int_{z(\ )>0} S\,\delta x\,dy + \int_{z(y)=0} S\delta z\,dy + \int_b^g [F_x - (F_{x'})']\,\delta x\,dy \quad (6.1)$$

The first two integrals are taken here along ab, and summation is carried out at all kink points, Δx and Δy are the coordinate increments of these points, n is the point of continuity of ξ' which is the point of reflection of discontinuity of multipliers $\mu_1, \ldots, \mu_4$ if there are several such points one more summation sign is to be added in (6.1)).

Coefficients $U^{(i)}$ and $V^{(i)}$, and S are defined by Formulas

$$U^{(1)} = (\Phi_{x'} + \alpha)_- - F_{x'+}, \qquad U^{(2)} = (\Phi - x'\Phi_{x'} - \alpha x')_- - F_+$$

$$V_j^{(1)} = (\Phi_{x'-} + \alpha_- - \Phi_{x'+} - \alpha_+)_j - \int_{j-}^{j+} \Big[\mu_1 v du + \mu_2\Big(v dv + \frac{1}{\rho}dp\Big) + \\ + \mu_3 d(\rho v) + \mu_4\Big(v dh - \frac{v}{\rho}dp\Big)\Big]$$

$$V_j^{(2)} = [(\Phi - x'\Phi_{x'} - \alpha x')_- - (\Phi - x'\Phi_{x'} - \alpha x')_+]_j + \int_{j-}^{j+} \Big[\mu_1\Big(u du + \frac{1}{\rho}dp\Big) + \\ + \mu_2 u dv + \mu_3 d(\rho u) + \mu_4\Big(u dh - \frac{u}{\rho}dp\Big)\Big]$$

$$S = \Phi_x - (\Phi_{x'})' - \alpha' + v(\xi'\mu_2 - \mu_1)u' - \rho\mu_3[(\rho u)' + \nu y^{-1}\rho v]$$

The integrals in the expression of $V_j^{(i)}$ are computed at the angle point, i.e, for values $y \equiv y_j$ and $x \equiv x_j$, and $x' = u/v$.

If the contour ag is an optional one (i.e. yields maximum χ), then in accordance with (6.1) and (5.3) it must satisfy the following necessary conditions

$$S = 0 \quad \text{for } z(y) > 0, \qquad S > 0 \quad \text{for } z(y) = 0 \ \text{(along } ab)$$

$$F_x - (F_{x'})' \geqslant 0 \ \text{(along } bg), \qquad U_b^{(1)} \geqslant 0, \quad U_b^{(2)} = 0$$

$$V_j^{(1)} = V_j^{(2)} = 0 \quad \text{for } z(y_j) > 0 \qquad (6.2)$$

$$V_j^{(1)}\zeta'(y_j) + V_j^{(2)} \geqslant 0,\ V^{(1)}{}_j \geqslant 0 \quad \text{for } z(y_j) = 0$$

$$[\mu_2]_{2n} = 0 \quad \text{for } (\xi_-' - \xi_+')_n = 0$$

in which the conditions at point b have been written for the case of $y_b < y_g$ only.

It is evident from the last condition of (6.2) that the second set characteristics which are lines of discontinuity of the Lagrange multipliers are in the case of an optimal contour either absent, or reach kink points of the wall contour, i.e. they belong to the corresponding rarefaction wave beams. Hence, in accordance with (5.1), (5.2) and (4.10) either $v_b = 0$, or n is an angle point (Fig. 1). This situation will repeat itself until a characteristic of the first set emanating from the next following angle point intersects characteristic ah. If $\Phi_x' \equiv 0$ which happens, for example, in the absence of isoperimetric conditions, then equality $v_b = \hat{0}$ cannot generally be fulfilled. At the same time two additional arbitrary selections correspond at each angle point to the two equalities appearing in (6.2) when $z(y_n) > 0$, viz., the x_n-coordinate, and the magnitude of the kink $(\xi'_+ - \xi'_-)_n$. Thus, if the closing characteristic of an optimal contour begins at the axis of symmetry outside of the initial rarefaction wave fan, and provided there are no shock waves in the influence area, then such a contour will have along section ab not less than one kink point (notations on Fig. 2 are the same as on Fig. 1).

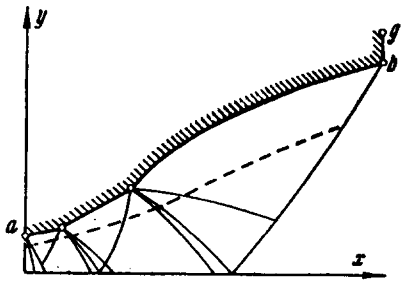

Fig. 2

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ON THE STABILITY OF A PLANE COUETTE FLOW

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Sufficient stability conditions (1.8), (1.10) are defined. Stability for large Reynolds numbers R is analyzed by asymptotic and numerical methods; it is shown that the flow is stable for $R \to \infty$

1. The stability of plane Couette flow is determined by the eigenvalues of the problem condidered in [1]

$$(D^2-\alpha^2)^2\varphi - i\alpha R(y-c)(D^2-\alpha^2)\varphi = 0 \qquad \left(D = \frac{d}{dy}\right) \tag{1.1}$$
$$D\varphi(\pm 1) = \varphi(\pm 1) = 0 \qquad (-1 \leqslant y \leqslant 1)$$

The flow is stable if for any values of the Reynolds number R and of the wave number α, all of the eigenvalues $c = c_r + ic_i$ have a negative imaginary part.

Investigators [2 – 8] of the problem (1.1) assumed the flow to be stable; this assumption had not been completely substantiated thus far, however, because either particular values of parameters R and α, or special eigenvalues only had been considered. The particular case of $R \to \infty$ is considered below, but in contrast to papers [2 and 5 – 7] only one of the quantities (*) $\varepsilon = (\alpha R)^{-1/3}$, $\delta = \alpha\varepsilon$

which express the eigenvalues is assumed to be small.

The characteristic relationship of problem (1.1) can be presented in the form [2]

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*) The case of small ε and arbitrary δ was inaccurately analyzed in [6], see [2 and 5].